3.3 – Orthogonality

Definition 1: Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$. We will also agree that the zero vector in R^n is orthogonal to every vector in R^n.

A vector $\mathbf{n}$ that is orthogonal to a line in R^2 or R^3 or a plane in R^3 is called a **normal**.

3. Find a point-normal form of the equation of the plane passing through P and having $\mathbf{n}$ as a normal.
$P(-1, 2, -1)$, $\mathbf{n} = (-2, 1, -1)$

Theorem 3.3.1

a) If a and b are constants that are not both zero, then an equation of the form
$ax + by + c = 0$ represents a line in R^2 with normal $\mathbf{n} = (a, b)$.

b) If a, b, and c are constants that are not all zero, then an equation of the form
$ax + by + cz + d = 0$ represents a plane in R^3 with normal $\mathbf{n} = (a, b, c)$.

Theorem 3.4.3 (not a typo)

If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Theorem 3.3.2 Projection Theorem

If $\mathbf{u}$ and $\mathbf{a}$ are vectors in R^n, and if $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

When $\mathbf{u}$, a vector in R^n, is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of a vector $\mathbf{a}$ in R^n, $\mathbf{w}_1 = \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$** or the **vector component of $\mathbf{u}$ along $\mathbf{a}$**. The vector $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$ is called the **vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$**.

19. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.
$\mathbf{u} = (2, 1, 1, 2)$, $\mathbf{a} = (4, -4, 2, -2)$

38. Find the standard matrix for the orthogonal projection of R^2 onto the stated line, and then use that matrix to find the orthogonal projection of the given point onto that line.

The orthogonal projection of $(1, 2)$ onto the line that makes an angle of $\pi/4$ $(= 45°)$ with the positive x-axis.

36. Find the standard matrix for the reflection of R^2 about the stated line, and then use that matrix to find the reflection of the given point about that line.

The reflection of $(1, 2)$ about the line that makes an angle of $\pi/4$ $(= 45°)$ with the positive x-axis.

Theorem 3.3.3 Theorem of Pythagoras in R^n

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in R^n with the Euclidean inner product, then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.